Crossing of identical solitary waves in a chain of elastic beads

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We consider a chain of elastic beads subjected to vanishingly weak loading conditions, i.e., the beads are barely in contact. The grains repel upon contact via the Hertz-type potential, $V \propto \delta^n$, $n>2$, where $\delta \geqslant 0$, δ being the grain–grain overlap. Our dynamical simulations build on several earlier studies by Nesterenko, Coste, and Sen and co-workers that have shown that an impulse propagates as a solitary wave of fixed spatial extent (dependent only upon n) through a chain of Hertzian beads and demonstrate, to our knowledge for the first time, that colliding solitary waves in the chain spawn a well-defined hierarchy of multiple secondary solitary waves, which is $\sim$ 0.5% of the energy of the original solitary waves. Our findings have interesting parallels with earlier observations by Rosenau and colleagues [P. Rosenau and J. M. Hyman, Phys. Rev. Lett. **70**, 564 (1993); P. Rosenau, *ibid.* **73**, 1737 (1994); Phys. Lett. A **211**, 265 (1996)] regarding colliding compactons. To the best of our knowledge, there is no formal theory that describes the dynamics associated with the formation of secondary solitary waves. Calculations suggest that the formation of secondary solitary waves may be a fundamental property of certain discrete systems.

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I. INTRODUCTION

The study of impulse propagation in a chain of coupled elastic beads can exhibit interesting nonlinear dynamics [1–4]. In the absence of external loading between the beads that are barely in contact, any impulse propagates as a solitary wave through a chain of elastic beads [1–8]. The presence of loading destroys the solitary wave [1–4,9], which becomes dispersive as a function of the magnitude of loading. The nature of dispersion is sensitive to the competition between the amplitude of the impulse and loading. Therefore, the no-loading case [8] is one of the most interesting regimes in which one can probe the nonlinear dynamical problem of impulse propagation in Hertzian beads. It turns out that in the absence of loading, any impulse, no matter how weak, generates solitary waves in the chain of elastic beads [10,11].

It is well known that acoustic signals, which can be obtained by sending an impulse in a chain of elastic beads under external loading [8,12], backscatter off buried inclusions in the chain of Hertzian beads. In the absence of such loading, solitary waves also backscatter off impurities or inclusions in Hertzian chains [13]. Backscattering of solitary waves from buried inclusions exhibit unusual behavior, as noted in Ref. [13]. In the present study, we focus on an important idealization of the problem of backscattering of solitary waves, namely, one in which two identical solitary waves traveling in opposite directions cross one another. This problem is also identical to that of backscattering of a solitary wave from an infinitely massive impurity at the center of a chain and hence may be relevant to the study of the general problem of backscattering of solitary waves from impurities.

This paper is arranged as follows. Section II presents the details of the model system and summarizes the details of the calculations. The calculations on the crossing of solitary waves in a chain of discrete beads reveal certain peculiar properties associated with the intersection of the solitary waves that are specific to Hertzian systems and are presented in Sec. III. We mention that there are possible connections between the results obtained in this work and those obtained by researchers in a variety of related systems, which are also known as compact solitary waves or compactons [14–19]. The calculations are further checked by carrying out extensive time-reversed dynamical studies. This is reported in Sec. IV. Preliminary studies on the hierarchy of solitary waves that spawn at the point of crossing of the solitary waves is discussed in Sec. V. The work is summarized in Sec. VI.

II. MODEL SYSTEM AND ANALYSES

We consider a linear system of macroscopic, monodisperse beads of mass m and radius R. We let E and σ denote the Young's modulus and the Poisson ratio, respectively. We assume that two such beads repel upon intimate contact according to Hertz law [20]. To proceed further, we define the overlap $\delta_{i,i+1} \equiv \Delta-(r_{i+1}-r_i)$, where r_i denotes the displacement of grain i from the equilibrium position and Δ is the initial loading at the grain–grain contact (compression at equilibrium). For spherical beads, the repulsive potential is given by

$$V(\delta_{i,i+1})=(2/5D)(R/2)^{0.5},$$

$$\delta_{i,i+1}{}^n \equiv a\,\delta_{i,i+1}{}^n \quad \text{if } \delta_{i,i+1} \geqslant 0, \tag{1}$$

$$\equiv 0 \quad \text{if } \delta_{i,i+1} < 0,$$

where $D \equiv 3/2(\{1-\sigma^2\}/E)$ and $n=5/2$. If $n=2$, the repulsive force is harmonic. In general, the constant a and the magnitude of n depend upon the contact geometry between the beads and typically varies between $n=5/2$ and 3 [21]. Due to the nonlinear nature of the repulsion between adjacent grains, one might expect that dynamical phenomena involving significant compression and decompression of grains would be highly nonlinear in nature.

Nesterenko [1–4] used a certain continuum approximation of the relevant equations of motion to show that an impulse initiated in a horizontally aligned system of beads would travel as a solitary wave when it is far from the boundary. In addition, Nesterenko showed that his solitary waves spanned some 5 bead diameters [1–4]. Nesterenko also carried out numerical studies on impulse propagation albeit with less accuracy [1] than what we shall see in the present work and reported experiments to demonstrate the propagating solitary waves [2,4]. Recently, high accuracy numerical work to improve upon Nesterenko's long wavelength approach has been carried out by Chatterjee [1]. Several experiments have since confirmed the presence of solitary waves in Hertzian systems [2,7,22]. It is important to note that Nesterenko and co-workers did not publish analytical, computational or experimental studies on the problem of crossing of two identical solitary waves. Recent work carried out by some of us have extended Nesterenko's theoretical studies and described the structure of solitary waves in systems of discrete beads [10]. To the best of our knowledge, the problem of crossing of solitary waves in Hertzian bead chains remains to be explored.

The equation of motion of some bead i (excluding the edge grains) in a finite chain of Hertzian beads is given by

$$md^2r_i/dt^2 = na\{[\Delta-(r_i-r_{i-1})]^{n-1} - [\Delta-(r_{i+1}-r_i)]^{n-1}\}. \tag{2}$$

Initially, every grain is placed barely in touch with one another such that $\Delta=0$. We call this the ''no-loading'' case. An impulse defined by an initial velocity v_0 at time $t=0$ is initiated at the first bead (i.e., at the boundary). As the impulse propagates, we find via numerical studies that a well-defined solitary wave develops in space and time. Typically, a series of solitary waves of much smaller amplitude are also generated by an arbitrary impulse. The details of the impulse generation process are intimately connected with that of solitary wave formation. Thus, impulses generated over modest time windows (compared to the natural period of the grains due to the Hertzian potential) can result in the generation of multiple solitary waves of hierarchical amplitudes. At present, much work remains to be done to understand the problem of impulse generation versus solitary wave formation [8,13]. We shall address this issue later in Sec. V.

Let us focus on the properties of the solitary wave itself. High precision (nine digit accuracy in energy calculations that demonstrate that energy conservation is accurately valid in our calculations) numerical study of Eq. (2) reveals that the solitary wave propagates at a fixed velocity c. Hence, the distance traveled by a solitary wave can be represented by $z=ct$, where t is the elapsed time since the initiation of the impulse. One can then rewrite Eq. (2) as follows:

$$mc^2\, d^2\varphi_n(z)/dz^2 = \{na[\varphi_n(z-2R)-\varphi_n(z)]^{n-1} - [\varphi_n(z)-\varphi_n(z+2R)]^{n-1}\}, \tag{3}$$

which is a convenient form for analyzing the solitary wave propagation problem. The subscript n in φ_n is to remind the reader that the properties of φ are sensitive to the contact geometry and hence on n [21,22]. In Eq. (3), we use for the solitary wave solution $\varphi_n(z-ct)=(r_i(t)-r_i(0))$ and $\varphi((z\pm 2R)-ct)=(r_{i\pm1}(t)-r_{i\pm1}(0))$.

The approximate analytic solution to Eq. (3), which is described in Ref. [8] and is in impressive agreement with the numerically generated solution, can be written as

$$\varphi_n(z) = \frac{A}{2}\left[1-\tanh\left(\frac{f_n(z)}{2}\right)\right], \tag{4}$$

where

$$f_n(z) = \sum_{q=0}^{\infty} C_{2q+1}(n) z^{2q+1}. \tag{5}$$

The coefficients $C_{2q+1}(n)$ depend on n only and can be determined by substituting Eq. (4) into Eq. (3), and minimizing the integral of the squared difference between the right-hand side and the left-hand side. For $n=5/2$, $C_1=2.3953$, $C_3=0.2685$, $C_5=0.006\,13$. The higher order coefficients are significantly smaller than C_5.

Our calculations show that for $n=5/2$, the solitary wave spans about 5 bead diameters [10,13] in perfect agreement with Nesterenko's theoretical and experimental findings [3]. Using our approach, we also recover that the speed of the solitary wave, $c \propto A^{1/4}$, where A is the amplitude of displacement of a bead due to the solitary wave.

The reader should note that *there is no exact continuum limit of the problem of impulse propagation in discrete Hertzian chains under conditions of no-loading*. Therefore, Nesterenko's analyses [1–4] and that of Chatterjee [1] cannot be directly compared with our work. In our work we have attempted to directly solve the equation of motion [Eq. (3)] by trial and error. The validation of our solution is presented via the excellent agreement between the numerically generated and the analytically generated solutions in Ref. [10]. The key difference between our solution and of the other researchers lie in the fact that we do not need to consider a portion of our solution to generate the solitary wave.

One may note that a similar problem has been studied for the case of $n=4$ (but not for a system of beads with no-loading where $\delta \geqslant 0$) by Kivshar in Ref. [18].

In our numerical calculations, $m=1$, $a=1$, $v_0=1$, and the integration time step is kept at 10^{-3}. The calculations have been carried out using the sixth order Gear predictor–corrector algorithm [20].

III. CROSSING OF IDENTICAL SOLITARY WAVES

We describe a study in which *two solitary waves* (as opposed to impulses, which invariably generate a hierarchy of

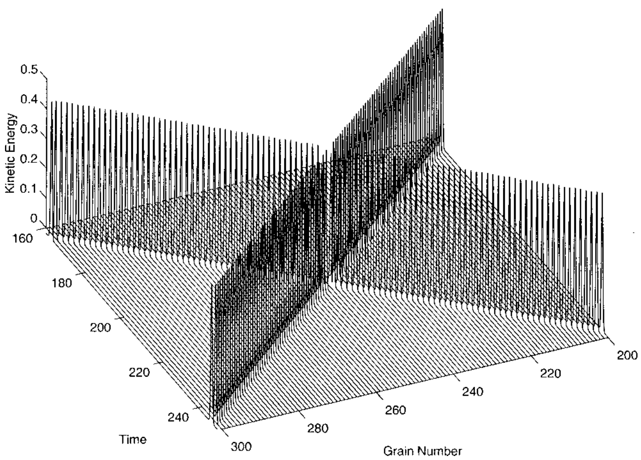

FIG. 1. Data shows the crossing of two identical solitary waves traveling in opposition in a chain of Hertzian beads. Kinetic energy of grain, time and grain positions are represented in linear order from chain ends and are all depicted in arbitrary units.

solitary waves) of same amplitude but opposite displacements are initiated at the two ends of a chain with an odd number of beads. The system is set up in such a way that the solitary waves intersect one another at the *center* of the central bead of the chain. It should be mentioned that the results of collision between solitary waves may be slightly different if they do not collide at the center bead but at an arbitrary point along the line joining a bead center to the contact point between two beads. We do not carry out a systematic study of such ''off-center'' collisions in this first study.

An important question to address is whether at the point of intersection, the opposing solitary waves ''cancel out,'' i.e., whether the center of the central bead suffers any motion at any time. In earlier numerical analyses of limited precision carried out by Nesterenko, it was found that the solitary waves underwent perfect annihilation at the point of crossing. Figure 1 shows a 3D plot of kinetic energy versus distance (measured in bead diameters) and time for half of the chain length. There appears to be no residual motion at the center of the central bead from the data in Fig. 1. As we shall see, more resolution of the data reveals that there is no motion of the central bead at any time and that there is significant motion of the elastic beads in the immediate vicinity of the central bead.

In Fig. 2, we present the same data with energy resolution that is 10^4 times greater than that in Fig. 1. As alluded to above, more scrutiny reveals that while there is no residual motion at the point of intersection, the adjacent beads begin to oscillate or ''rattle.'' This behavior is evident upon observing the temporal behavior of grains 249 and 251 and those beyond in either direction. Such rattling gives rise to the generation of multiple solitary waves of progressively diminishing amplitudes, which move at progressively slower velocities. We call these waves, ''secondary'' solitary waves.

Figure 3 describes the temporal displacements of a few of the neighbors of the central bead where the solitary waves cross. The calculations suggest that secondary solitary waves with progressively weakening amplitudes would be generated by neighbors that are farther removed from the central bead. The residual motion typically involves some 0.5% of the total kinetic energy of the two solitary waves. Not surprisingly, the hierarchy of secondary solitary waves, as reported here, was not observed in earlier calculations [17].

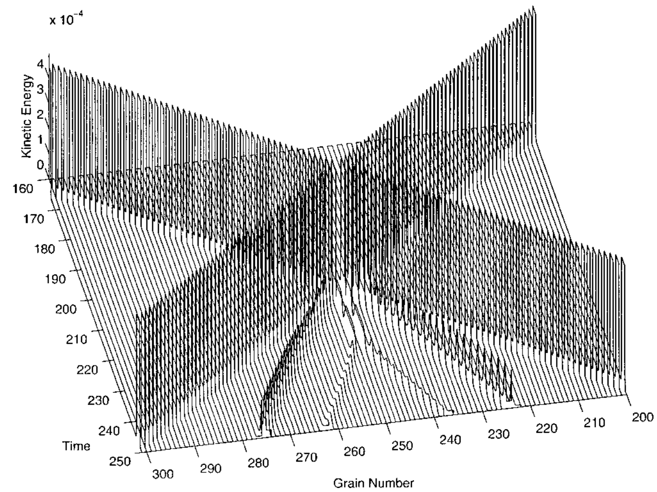

FIG. 2. The plot shows that smaller solitary waves of progressively decaying amplitude and velocity are created after the intersection of the solitary waves, which possess energies that are about 10^{-4} of the original solitary waves. Observe that only a part of the original solitary waves is visible. Kinetic energy of grain, time and grain positions are represented in linear order from chain ends and are all depicted in arbitrary units.

IV. SOLITARY WAVE FORMATION IN TIME-REVERSED DYNAMICS

An important issue to address is whether the magnitudes of the secondary waves are comparable to the unavoidable numerical errors that are incurred during a numerical calcu-

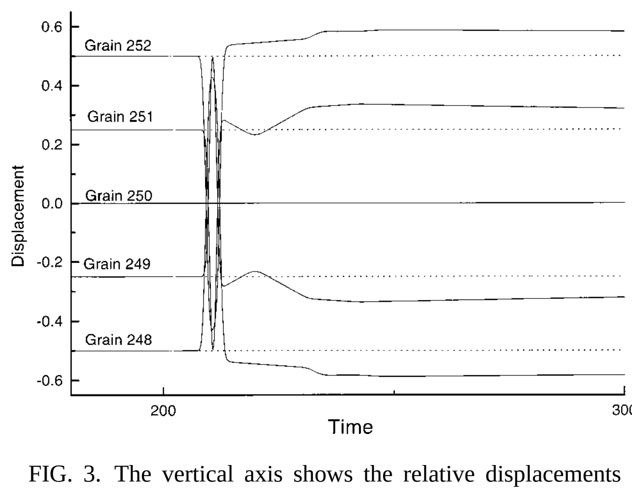

FIG. 3. The vertical axis shows the relative displacements of beads (in arbitrary units) adjacent to grain number 250 where collision occurs as function of time (in arbitrary units). The dashed lines represent the equilibrium positions of the beads. Each horizontal solid line shows the displacement of a bead with respect to its original equilibrium position. Observe that the central grain does not move in time.

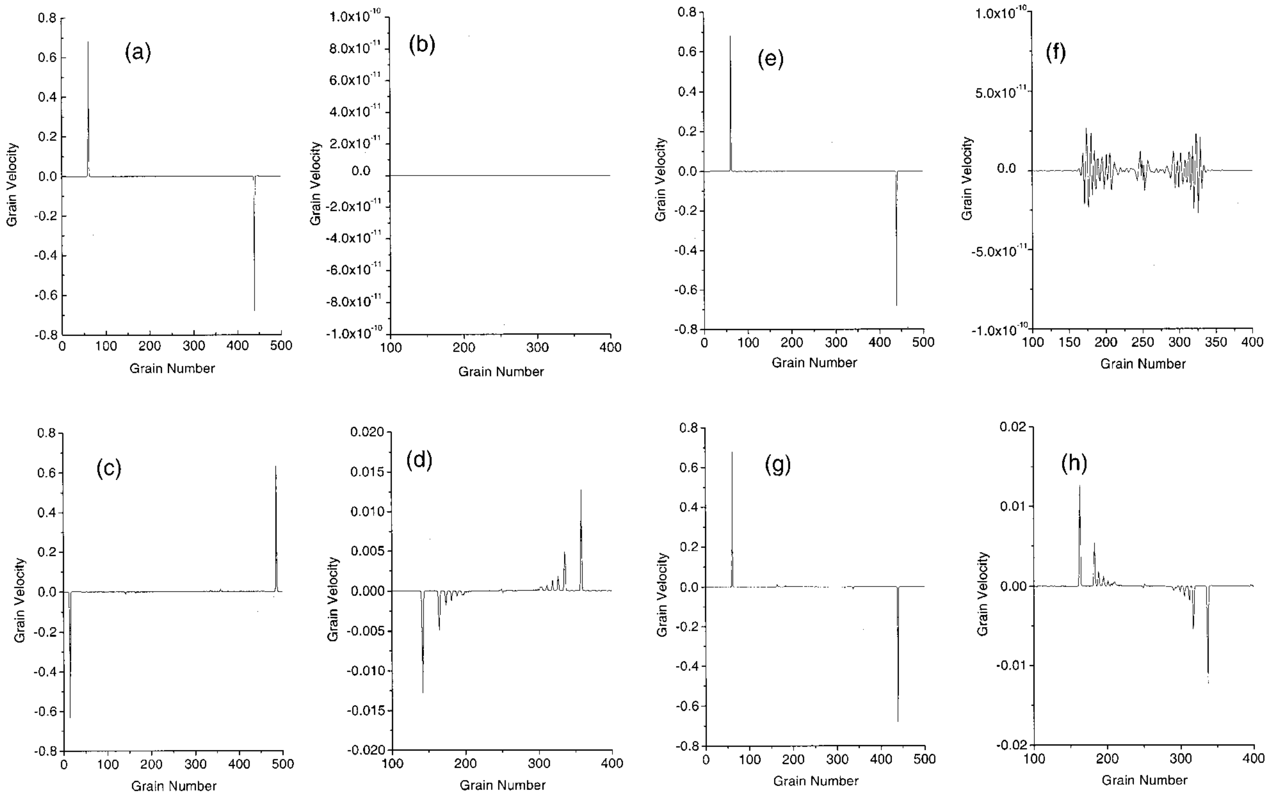

FIG. 4. Figures show velocities (in arbitrary units) of grains versus grain positions in linear order from chain ends (in arbitrary units). (a) Initial system, with two solitary waves moving toward each other. (b) The rest of the chain is initially at rest. (c) After solitary waves cross, some energy remains behind in the chain. (d) A magnification of *y* axes shows than this energy propagates as secondary solitary waves. (e) After running the system in (c) back in time, a system almost identical to that in (a) is obtained. (f) The small differences between the systems are due to numerical errors. Please observe the order of magnitude of the noise. (g) Same as in (e) but now the secondary solitary waves in the system of (c) are removed before the time-reversal procedure. (h) Secondary solitary waves are again obtained.

lation. Figure 4(a) shows a snapshot of two equal solitary waves propagating toward each other. Figure 4(b), present data with significantly higher resolution that reveals there is no motion of the grains located between the positions of the incoming pulses at the instant of time at which Fig. 4(a) has been recorded.

In Fig. 4(c) we present a snapshot at a time instant that records the grain dynamics after the pulses have crossed each other. Although the pulses appear to remain undistorted after traveling through one another, it can be seen that some small energy bundles, trapped as secondary solitary wave pulses, seem to lag the main pulses (in the region between the main pulses). In Fig. 4(d) we magnify the data along the vertical axis by a factor of 100, such that the secondary waves become visible. These figures reveal that the crossing of solitary waves in discrete chains leads to the formation of secondary solitary waves.

Since the Newtonian dynamics of a system should remain unchanged under time-reversal, we integrated the system shown in Fig. 4(c) backward in time and obtained Fig. 4(e). In Fig. 4(e), the system has been restored to the state captured in Fig. 4(a). As in any calculation, our numerical integration causes errors. The magnitudes of such errors are recorded in Fig. 4(f). The reader may observe that the ''noise'' generated by numerical errors is many orders of magnitude smaller than the secondary solitary waves.

Another test of numerical accuracy was done as follows. We started our numerical simulations with the system in the state represented in Fig. 4(c) except that we kept only the main pulses and reset to zero the secondary waves between pulses. Observe that the solitary waves are about to collide. Hence, the directions of propagation of the solitary waves in Fig. 4(g) are reversed with regard to the same in Fig. 4(c).

The main pulses (which now behave as independent solitary waves) again generate secondary waves upon collision. These pulses are shown in Fig. 4(h) and closely resemble the secondary solitary waves shown in Fig. 4(d). We conclude that the secondary waves generated after the collision of two solitary waves are not due to the errors of the numerical integration.

V. HIERARCHY OF SECONDARY SOLITARY WAVES

As shown in Figs. 2 and 3, the secondary solitary waves possess much smaller amplitudes and hence move much slower compared to the original solitary waves. Their small magnitudes make numerical analysis of dynamics of these objects rather challenging. To probe the time evolution of the secondary solitary waves, one must not only carry out simulations over significantly longer times, but also with higher precision. In the reported data, the precision associated with the kinetic energies of beads is at least 4 orders of magnitude larger than the accuracies at which round-off errors are incurred. We have carried out our analyses using the maximum accuracy that we can accommodate and yet maintain reasonable calculation times [23].

Our dynamical simulations *suggest* that the formation of the secondary solitary waves is a consequence of the discrete nature of the beads in the Hertzian system. In the continuum approximation, that is invoked in the long wavelength regime, the details of motion of grains that are located adjacent to the point of intersection of solitary waves are coarse grained and hence may even disappear. Therefore, in existing theoretical analyses of dynamics of Hertzian systems, secondary solitary waves may not appear at all.

An interesting question that arises is whether secondary solitary waves should be expected to exist in all other discrete systems that support solitary waves of finite size and what role does the size of the solitary wave play in eliminating secondary solitary waves. Our preliminary calculations suggest that secondary solitary waves may not necessarily arise in all discrete systems. However, given the difficulty level of this problem, further analysis is necessary [24] before one can draw definitive conclusions about which discrete systems, in general, will admit secondary solitary waves.

Figure 5(a) shows the maximum displacement, velocity and acceleration of the secondary solitary waves as a function of the order of the solitary waves. Given the rapid decay in the magnitudes of the quantities measured with respect to the order of secondary solitary wave and the fact that the decay looks rather similar when plotted using log–log [shown in Fig. 5(a)] and semi-log scales, it is also difficult to draw definitive conclusions about the pattern associated with the decay.

To test our intuition about the formation of secondary solitary waves at or near an interface, we searched for the spawning of secondary solitary waves formed immediately after an impulse has been imparted to an end of a chain of Hertzian beads. Our results are shown in Fig. 5(b). Once again, we find that the maximum displacement, velocity and acceleration of secondary solitary waves decay with increasing order of the solitary wave.

To the best of our knowledge, this work presents the first systematic study that suggests that any impulse generated on a chain of Hertzian grains is likely to ''quantize'' itself into successive solitary waves of fixed width but progressively decaying amplitude at a boundary.

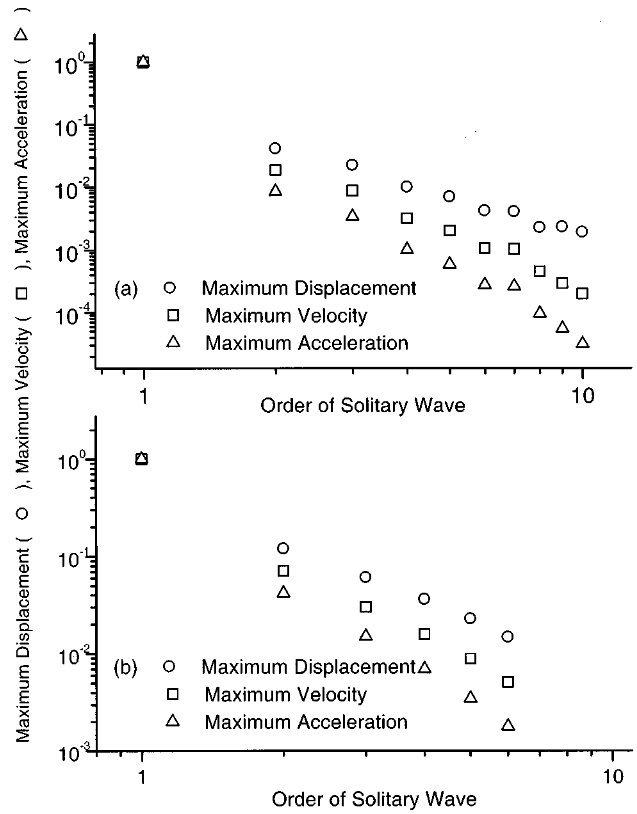

FIG. 5. (a) Maximum displacement, velocity and acceleration (all in arbitrary units) of the secondary solitary waves that are formed at the point of crossing. (b) Maximum displacement, velocity and acceleration (all in arbitrary units) of the secondary solitary waves that form at the boundary at which the impulse is initiated.

VI. SUMMARY AND CONCLUSIONS

In closing, we have presented a numerical study of the equation of motion that describes the dynamics of individual grains in a chain of Hertzian beads with zero loading. We confirm that an impulse propagates as a solitary wave in a chain of Hertzian grains. We further demonstrate that the crossing of identical solitary waves in chains of discrete Hertzian grains lead to the spawning of weak secondary solitary waves at the point of intersection of the solitary waves. The hierarchy of magnitudes of these secondary solitary waves appears to depend upon the details of the Hertz law [20].

We hope that the present contribution will stimulate detailed experimental analyses of secondary solitary waves.

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